J. Fluid Mech. (1973), vol. 59, part 2, pp. 397–413
Printed in Great Britain

Resonant surface waves

By J. R. OCKENDON AND H. OCKENDON
Mathematical Institute, Oxford

(Received 12 September 1972)

Small amplitude forced horizontal or vertical oscillations of a container of liquid with a free surface may give rise to motions in the liquid on a scale much greater than the forcing amplitude. Three such situations are analysed and, in those cases where the response is still small compared with the dimensions of the container, explicit asymptotic solutions for the liquid motion are obtained.

1. Introduction

This paper is concerned with three types of gravity waves which form on the surface of a liquid in a container which oscillates at near-resonant frequencies. By resonant oscillations we shall mean that the response to forced harmonic oscillations whose amplitude is small compared with the dimensions of the container is of a larger order of magnitude than the forcing amplitude. The most amenable class of such oscillations is that in which the response is still small compared with the dimensions of the container and the first two problems considered fall into this category. Although the three problems are physically distinct, as perturbation problems they are very similar. They all involve at least two small parameters in such a way that a linear description becomes invalid as one of the parameters approaches zero. In each case there is a critical relationship between the orders of magnitude of the parameters which, when it holds, leads to a nonlinear asymptotic solution of the problem which remains valid as any of the parameters tends to zero.

For simplicity we shall discuss oscillations of inviscid fluids initially and subsequently consider situations in which laminar viscous boundary layers on the container walls can have an important effect.

The first problem, which has been fairly thoroughly discussed in the literature (Moiseyev 1958; Chester 1968), concerns horizontal oscillations of a two-dimensional finite tank of length πL and depth hL. Free oscillations have a discrete spectrum with frequencies $\omega = [gn \tanh (nh)/L]^{\frac{1}{2}}$, where n is an integer, and resonance occurs when the tank is forced to oscillate near these frequencies. In § 2 we use the asymptotic methods described above to obtain a uniformly valid description of the response which includes the results of both Moiseyev and Chester and unifies their analyses in the case of weak dispersion when h is small.

The second problem, which is not so well understood, deals with horizontally forced oscillations in a semi-infinite tank of breadth πb and depth hb. Here there is a continuous spectrum of free oscillations for any given transverse wavenumber m, with frequencies

$$\omega = \{g(m^2+\sigma^2)^{\frac{1}{2}} \tanh [(m^2+\sigma^2)^{\frac{1}{2}} h]/b\}^{\frac{1}{2}},$$

where σ is real. Resonance does not occur at all these frequencies (Ursell 1952) but only at the cut-off frequency $\omega_0 = [gm \tanh (mh)/b]^{\frac{1}{2}}$. An interesting feature of this situation is that the linearized solution is not only singular as $\omega \to \omega_0$ but the boundary condition at infinity needed to specify the solution uniquely changes from one of boundedness for $\omega < \omega_0$ to a radiation condition for $\omega > \omega_0$. The equations which describe the transition region when $|(\omega - \omega_0)/\omega_0|$ is small have been derived by Mahony (1971) and § 3 is mainly concerned with a discussion of the possible solutions of these equations.

The final problem concerns vertical oscillations of a cylindrical tank of arbitrary cross-section. The response is no longer one of classical resonance, since a rigid-body motion of the fluid is a solution of the full, as well as the linearized, equations of motion for all forcing frequencies. However, near the discrete spectrum of natural frequencies of the tank, Benjamin & Ursell (1954) have shown that this solution is unstable and in § 4 we present a theory which suggests that, for frequencies above the lowest natural frequency, motions are possible on a length scale comparable with that of the container, even for infinitesimal forcing amplitudes.

Our aim in each problem is to discover the response curve for the amplitude of periodic solutions in terms of the fixed forcing frequency. For the third problem it is relatively easy to discuss the stability of the various branches of the response curve. Elsewhere we shall only touch upon the problems of stability and of the way in which the response curve is traced out when the forcing frequency is varied sufficiently slowly.

2. Horizontal oscillations of a finite tank

The inviscid two-dimensional motion of an incompressible fluid of mean depth hL in a tank of length πL subjected to a horizontal oscillation of amplitude ϵL and frequency ω is described by a velocity potential $\epsilon L^2 \omega \phi$ and surface elevation $\epsilon L \eta$ satisfying, with suitably chosen axes,

$$\nabla^2 \phi = 0, \tag{2.1}$$

$$\phi_x = \sin t \quad \text{on} \quad x = -\epsilon \cos t, \pi - \epsilon \cos t \tag{2.2}$$

and, on $y = \epsilon\eta$,

$$\phi_y = \eta_t + \epsilon \phi_x \eta_x, \tag{2.3}$$

$$\eta + (1+\delta) \tanh h[\phi_t + \tfrac{1}{2}\epsilon(\phi_x^2 + \phi_y^2)] = 0 \tag{2.4}$$

in dimensionless variables. Here $L\omega^2 = (1+\delta) g \tanh h$. This section will be concerned with solutions for ϕ and η which are periodic with the same frequency as the forcing term, η having zero mean. The first resonant frequency according to linearized theory, when $\epsilon = 0$, occurs when $\delta = 0$ and we shall only consider the solution for small ϵ and δ. The method we use for treating this situation may also be applied to higher resonant frequencies, but not uniformly for very high-frequency forcing.

An asymptotic expansion of the solution for small ϵ and δ yields the following results.

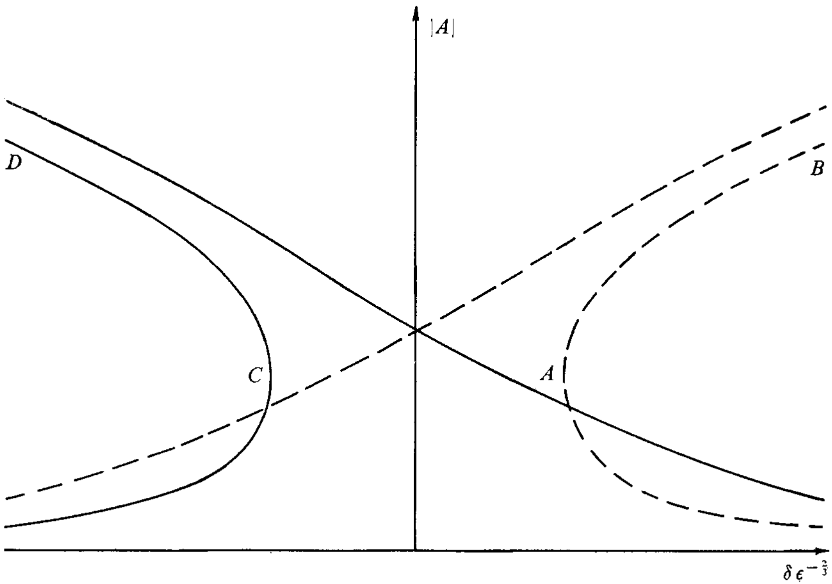

FIGURE 1. Response curves for equation (2.6). ——, $H > 0$; - - -, $H < 0$.

(i) $h \geqslant O(1)$. As long as $\delta \gg \epsilon^{\frac{2}{3}}$, the expansions for ϕ and η begin with terms of $O(\delta^{-1})$ and are the limiting forms of the linearized solution as $\delta \to 0$. The leading terms are the linearized eigensolutions

$$\phi \sim \frac{A}{\delta}\cos x \cosh(y+h)\sin(t+\alpha), \quad \eta \sim -\frac{A}{\delta}\sinh h \cos x \cos(t+\alpha) \qquad (2.5a, b)$$

and examination of the terms of $O(1)$ gives $\alpha = 0$ or π and $\pm 4/\pi \cosh h = A$ respectively. However, for $\delta\epsilon^{-\frac{2}{3}} \leqslant O(1)$ Moiseyev (1958) has shown that the appropriate expansions are $\phi \sim \epsilon^{-\frac{2}{3}}\phi_0 + \epsilon^{-\frac{1}{3}}\phi_1 + \phi_2 + \ldots$, and similarly for η. The motivation for this scheme on the assumption that ϕ_0 is still the linearized eigensolution (2.5a) is that, in order that the equation for ϕ_2, which must satisfy the boundary condition (2.2), should be soluble, it must contain a suitable odd harmonic in t and x generated by the nonlinear combination of the preceding terms. With the quadratic nonlinearity in (2.3) and (2.4), this is the simplest scheme which allows this to occur. After some manipulation we again find $\alpha = 0$ or π but now

$$\pm 4\pi^{-1}\tanh h = \delta\epsilon^{-\frac{2}{3}}A \sinh h + H(h)A^3 \qquad (2.6)$$

respectively, where $H = -\frac{1}{32}\operatorname{sech}^2 h \operatorname{cosech} h[9 + 15\sinh^2 h - 8\sinh^6 h]$ and is monotonic increasing with a zero at $h = h_0 \simeq 1{\cdot}06$. The response (figure 1) is thus exactly similar to that of an undamped Duffing equation with the change from 'hard-spring' to 'soft-spring' as h increases through h_0, and this phenomenon for free oscillations has been noted by Tadjbakhsh & Keller (1960).

A further property in common with Duffing's equation which may be easily verified is that of the instability of the branches AB or CD of the response curve. This may be done by considering a nearly periodic response in which A and α in (2.5) are functions of a slow variable $\tau = \epsilon^{\frac{2}{3}}t$ and using a two-variable expansion in t and τ. We also note that, when $h = h_0$, the expansion again breaks down for sufficiently small δ. However, for $|h-h_0| \leqslant O(\epsilon^{\frac{2}{5}})$, ϕ and η may be expanded in powers of $\epsilon^{\frac{1}{5}}$, with leading terms of $O(\epsilon^{-\frac{1}{5}})$, as long as $\delta\epsilon^{-\frac{4}{5}} \leqslant O(1)$. We shall not discuss the details here but the result is a quintic equation for A instead of (2.6) which gives a finite response for all finite δ and $h-h_0$.

These phenomena have been observed experimentally by Taylor (1953) and Fultz (1962). The experimental response curves naturally resemble those of a damped Duffing equation, but we shall not discuss in detail the various possible damping mechanisms for surface waves in finite containers here (see Miles 1967). However, we note that the effect of the linear oscillating boundary layers on the vertical walls of the tank is to replace (2.2) by

$$\phi_x = \sin t \pm (A/\delta Re^{\frac{1}{2}}) \cosh (y+h) \cos (t+\tfrac{1}{4}\pi) \tag{2.7}$$

on $x = -\epsilon \cos t, \pi - \epsilon \cos t$ respectively. Here $Re = \omega L^2/\nu$, where ν is the kinematic viscosity. Thus, if $Re^{\frac{1}{2}}\epsilon^{\frac{2}{3}} \sim O(1)$ there is a phase change in the boundary condition for ϕ_2 which results in a rounding off of the response curve for $|A|$. This is similar to the effect of introducing damping proportional to the velocity in Duffing's equation.

(ii) $h \ll 1$. This situation has been considered in some detail by Chester (1968) on the basis of the shallow-water-theory equations, with terms added to account for the effects of dispersion and viscous dissipation in the boundary layer at the base of the tank. For negligible dispersion and dissipation, the analogy with gasdynamics and the results for resonance in organ pipes (Chester 1964; Collins 1971) suggest that, as $\delta \to 0$, the most physically sensible solution for η is discontinuous, and Chester was able to consider the effects of dispersion on this solution. Indeed, the perturbation solution which we shall now describe, which connects the non-dispersive shallow-water limit with the limit as $h \to 0$ of the solution discussed above, will be found to satisfy the differential equation which was suggested by Chester as a model for dispersive effects in resonant oscillations.

The asymptotic expansion is now in terms of three small parameters ϵ, δ and h but we shall find that all the interesting cases can be discussed by considering the limit $\epsilon \to 0$ with $\kappa = h\epsilon^{-\frac{1}{4}}$ and $\lambda = \delta\epsilon^{-\frac{1}{2}}$ both of $O(1)$. We again construct a perturbation scheme in which the terms of $O(1)$ in the expansion for ϕ are capable of satisfying (2.2) and we find that this necessitates

$$\phi \sim \epsilon^{-\frac{1}{2}}\phi_0 + \epsilon^{-\frac{1}{4}}\phi_1 + \phi_2 + \dots, \quad \eta \sim \epsilon^{-\frac{1}{4}}\eta_0 + \eta_1 + \epsilon^{\frac{1}{4}}\eta_2 + \dots.$$

Thus ϕ_0 is a harmonic function whose normal derivatives vanish on $x = 0, \pi$ and $y = 0$ and the equation for ϕ_1 then shows that $\phi_{0yy} + \phi_{0tt} = 0$ on $y = 0$. Thus

$$\phi_0|_{y=0} = f_- + f_+, \tag{2.8}$$

where $f_\pm = f(t \pm x)$ and f has period 2π. A similar argument applied to ϕ_1, which is also harmonic, then gives

$$\phi_{1y}|_{y=0} = -\kappa(f''_- + f''_+). \tag{2.9}$$

Equations (2.8) and (2.9) are all we need to determine $\phi_{2tt} - \phi_{2xx}$ on $y = 0$ as

$$\tfrac{1}{3}\kappa^2(f^{\text{iv}}_- + f^{\text{iv}}_+ + f''_- + f''_+) - 3(f'_+f''_+ + f'_-f''_-) + f'_+f''_- + f'_-f''_+ - \lambda(f''_- + f''_+). \tag{2.10}$$

A particular integral for $\phi_2|_{y=0}$ can now be written down and ϕ_{2x} can only satisfy (2.2) if

$$2\sin t/\pi = (\lambda - \tfrac{1}{3}\kappa^2)f'' - \tfrac{1}{3}\kappa^2 f^{\text{iv}} + 3f'f''$$

or, if $f' = g$,

$$\frac{\kappa^2}{3}(g''+g) - \lambda g - \tfrac{3}{2}g^2 + \frac{2}{\pi}\cos t = -\frac{3}{4\pi}\int_0^{2\pi} g^2(t')\,dt'. \tag{2.11}$$

Here the constant of integration is chosen so that g and hence η_0 have zero mean over a period 2π.

We can now see in what sense this asymptotic solution covers all the interesting situations for small ϵ, δ and h. Away from resonance, as $\lambda \to \infty$, one asymptotic form for g is $2\cos t/\lambda\pi$, which is the limit of the linearized solution near resonance. For deeper water, as $\kappa \to \infty$, the effect of the nonlinearity is small and a perturbation solution for $\lambda \sim \kappa^{-\frac{2}{3}}$ gives $g \sim \kappa^{\frac{2}{3}} A\cos(t+\alpha)$, where A satisfies the limit of (2.6) as $h \to 0$. For shallower water, when $\kappa \to 0$, dispersion is small and one asymptotic form for g satisfies the algebraic equation (2.11) with $\kappa = 0$. The solution is described by Chester (1968) and compressive discontinuities occur for $|\lambda| < (96/\pi^3)^{\frac{1}{2}} \simeq 1{\cdot}76$.

Various aspects of the asymptotic and numerical solution of (2.11) are discussed in appendix A. We shall conclude this section with a brief account of the possible effect of viscosity in the boundary layer at the base of the tank. Although, in our scaling, $\phi \sim \epsilon^{-\frac{1}{2}}$ in the inviscid theory, the scale of motion is still small compared with the length of the tank. Thus the boundary layer is linear and the normal velocity at its edge is of $O(Re^{-\frac{1}{2}}\epsilon^{-\frac{1}{2}})$. In order for this correction to affect the equation for g we must have $(\epsilon Re)^{-\frac{1}{2}} \sim \epsilon^{\frac{1}{2}}$ or $Re \sim \epsilon^{-\frac{3}{2}}$, remembering that g is determined by (2.10), which is in turn derived from consideration of the term $\epsilon^{\frac{1}{2}}\phi_3$ in the expansion for ϕ. Now the free-stream velocity for the linear boundary layer is $\epsilon^{-\frac{1}{2}}(f'_+ - f'_-) = F(x,t)$ say. Thus the transverse velocity in the layer is

$$Re^{-\frac{1}{2}}\left[-\tilde{y}F_x - \pi^{-\frac{1}{2}}\int_0^\infty F_x(x,t-u)\left(\exp\left[-\tilde{y}^2/4u\right]-1\right)\frac{du}{u^{\frac{1}{2}}}\right], \tag{2.12}$$

where $\tilde{y} = (y+h)Re^{\frac{1}{2}}$. The resulting displacement effect means that the normal velocity of the outer flow tends to

$$\bar{\mu}\epsilon^{\frac{1}{2}}\int_0^\infty [f''(t-u+x)+f''(t-u-x)]\frac{du}{u^{\frac{1}{2}}}$$

as $y \to -h$, where $\bar{\mu} = \pi^{-\frac{1}{2}}\epsilon^{-\frac{3}{4}}Re^{-\frac{1}{2}}$. This finally means that the right-hand side of (2.11) is replaced by

$$-\frac{\bar{\mu}}{\kappa\pi^{\frac{1}{2}}}\int_0^\infty g(t-u)\frac{du}{u^{\frac{1}{2}}} + \text{constant}. \tag{2.13}$$

This term means that g is no longer symmetrical about $t = \pi$, and its effect on the numerical solution is discussed in appendix A.

3. Horizontal oscillations of a semi-infinite tank

We shall first consider the inviscid fluid motion described by Ursell (1952) in which a flexible wall oscillating with frequency ω in the plane $x = 0$ generates waves propagating in the positive x direction in a semi-infinite tank of breadth

πb and depth hb, the mean surface level being $y = 0$. Using dimensionless variables similar to those in § 2, the motion is described by

$$\nabla^2\phi = 0, \tag{3.1}$$

$$\phi_x = U(y)\cos z \sin t \quad \text{on} \quad x = -\epsilon U(y)\cos z\cos t, \tag{3.2}$$

$$\phi_y = 0 \quad \text{on} \quad y = -h, \tag{3.3}$$

$$\phi_z = 0 \quad \text{on} \quad z = 0, \pi \tag{3.4}$$

and, on $y = \epsilon\eta$,

$$\left.\begin{aligned} \eta + (b\omega^2/g)(\phi_t + \tfrac{1}{2}\epsilon|\nabla\phi|^2) &= 0, \\ \phi_y = \eta_t + \epsilon(\phi_x\eta_x + \phi_z\eta_z). \end{aligned}\right\} \tag{3.5}$$

Here $\epsilon b U(y)\cos z$ describes the spatial variation of the wall. Again we shall seek solutions for ϕ and η which are periodic in t with period 2π, η having zero mean. The appropriate boundary conditions as $x \to \infty$ will be discussed shortly.

When $\epsilon = 0$, the linearized solution for ϕ is proportional to $e^{\pm it}$ and can be expanded in a series of orthogonal eigenfunctions of the form

$$\exp[\pm i(k_0^2-1)^{\frac{1}{2}}x]\cosh k_0(y+h) \quad \text{and} \quad \exp[\pm(k_n'^2+1)^{\frac{1}{2}}x]\cos k_n'(y+h),$$

where $k\tanh kh = b\omega^2/g$ has roots $\pm k_0$, $\pm ik_n'$. If ω is less than the cut-off frequency $\omega_0 = [g\tanh h/b]^{\frac{1}{2}}$, so that k_0 is less than unity, the solution is determined uniquely by a boundedness condition as $x \to \infty$, and in fact it decays exponentially for large x. However, if $\omega > \omega_0$, a non-decaying wave train propagates to infinity and a radiation condition must be applied to ensure uniqueness. Moreover, if $\omega = \omega_0$, the boundary condition (3.2) cannot be expanded in terms of the eigenfunctions, which are no longer complete at $x = 0$. The amplitude of the linearized solution is then of $O(|\delta'|^{-\frac{1}{2}})$, where $k_0 = 1+\delta'$, and this section will be concerned with the solution of (3.1)–(3.5) for small values of ϵ and δ'.

As long as $k = \delta'/\epsilon > O(1)$, the asymptotic expansions for ϕ and η may be found from linearized theory, and initially proceed in powers of $|\delta'|^{\frac{1}{2}}$ with leading terms of $O(|\delta'|^{-\frac{1}{2}})$. A feature which distinguishes the expansions from those in § 2 is the appearance of two length scales, x of $O(1)$ and $O(|\delta'|^{-\frac{1}{2}})$, in the various terms of the solution. Mahony (1971) has shown that the appropriate expansions for $k \leqslant O(1)$ are $\phi \sim \epsilon^{-\frac{1}{2}}\phi_0 + \phi_1 + \epsilon^{\frac{1}{2}}\phi_2 + \ldots$, and similarly for η, where ϕ_i and η_i are functions not only of x, y, z and t but also of $\xi = x\epsilon^{\frac{1}{2}}$. As in § 2, ϕ_0 is again the linearized eigensolution

$$(A_0(\xi)\cos t + B_0(\xi)\sin t)\cos z\cosh(y+h) \tag{3.6}$$

but, in contrast, ϕ_1 can now satisfy (3.2) just as long as

$$\frac{dA_0}{d\xi} = 0, \quad \frac{dB_0}{d\xi} = -4\int_0^{-h} U(y)\cosh(y+h)\,dy/(\sinh h + 2h) = V, \tag{3.7}$$

say, at $\xi = 0$. A_0 and B_0 are not now determined until the solvability of the equation for ϕ_2 has been considered, because the introduction of the length scale ξ

introduces a resonant driving term $\epsilon^{\frac{1}{2}}\phi_{0\xi\xi}$ in the expansion of Laplace's equation. After some manipulation, we find that there is no bounded solution for ϕ_2 unless

$$d^2A_0/d\xi^2 + 2kA_0 + G(h)\,A_0(A_0^2+B_0^2) = 0, \qquad (3.8a)$$

$$d^2B_0/d\xi^2 + 2kB_0 + G(h)\,B_0(A_0^2+B_0^2) = 0, \qquad (3.8b)$$

where $G(h) = 2H(h)/\cosh h(\tanh h + h\,\mathrm{sech}^2 h)$.

A solution of (3.8) satisfying (3.7) is now sought, where we assume $V \neq 0$. In the linear problem, when $|k| \to \infty$, a unique solution exists satisfying the condition of boundedness as $k \to -\infty$ or the radiation condition

$$A_0 \cos t + B_0 \sin t \sim \text{constant} \times \exp\left[i((2k)^{\frac{1}{2}}\xi - t)\right] \quad \text{as} \quad k \to +\infty.$$

Each of these conditions can be thought of as the analytic continuation of the other in the complex-k plane. However, the enumeration of the solutions is more complicated in the nonlinear case, as may be seen by writing

$$A_0 + iB_0 = r\,e^{i\theta}, \quad r^2 = X, \quad r^2\,d\theta/d\xi = Y, \quad r\,dr/d\xi = Z$$

to give

$$dX/d\xi = 2Z, \quad dY/d\xi = 0, \quad X\,dZ/d\xi = Y^2 + Z^2 - GX^3 - 2kX^2, \qquad (3.9a,b,c)$$

with $Y^2 + Z^2 = V^2X$ at $\xi = 0$. This formulation is used for subsequent convenience. Consideration of the singular points in the X, Z phase plane with $X > 0$ and $Y = Y_0 =$ constant enables us to list the circumstances under which $X \to$ constant as $\xi \to \infty$, so that the radiation condition is satisfied.

First consider $Y_0 \neq 0$ and put $X = X_0$ when $\xi = 0$. Then, if $G > 0$, X can only tend to a constant if $X \equiv X_0$, where X_0 is the positive root of $GX_0^2 + 2kX_0 - V^2 = 0$. However, if $G < 0$ and $k > 0$, X can be identically equal to either root of this equation as long as $k^2 + GV^2 > 0$, but it can also tend to the larger positive zero of $GX^3 + 2kX^2 - Y_0^2$ as $\xi \to \infty$. Second, if $Y_0 = 0$, X may tend to zero as $\xi \to \infty$ if $k < 0$ and $GX_0^2 + 4kX_0 + 2V^2 = 0$, while if $k > 0$ and $G < 0$, X may tend to $-2k/G$ as $\xi \to \infty$ if $GX_0^2 + 4kX_0 + 4k^2/G + 2V^2 = 0$. Thus there are now many response curves for X_0, which is essentially the wave amplitude at the wave maker, as a function of k, and in particular there is an infinite spectrum when $G < 0$ and $k > 0$ (figures 2a, b).

This non-uniqueness may possibly be resolved by considering the stability of the various solutions, but instead we shall consider here the effect of suitably thin viscous boundary layers on the base and walls of the tank. Equation (3.6) shows that the velocity in these layers is predominantly in the y, z plane. Their analysis, when the Reynolds number is just small enough to affect the equations for A_0 and B_0, has been given by Mahony (1971) and is similar to the boundary-layer analyses given in § 2. The result is that, for $Re \sim \epsilon^{-2}$, the right-hand sides of (3.8a) and (3.8b) become $-\gamma B_0$ and γA_0 respectively, where γ is some measure of the viscosity, and also k is changed by $O(\gamma)$. We shall henceforth assume that $\gamma > 0$; our results are trivially modified for $\gamma < 0$. Equations (3.9) are thus basically unaltered except that the right-hand side of (3.9b) is now γX, so that the only singular point in the X, Y, Z phase space is the origin.

We shall now carry out our analysis of the viscous situation under the assumption of boundedness at infinity. A local linear analysis near $X = Y = Z = 0$

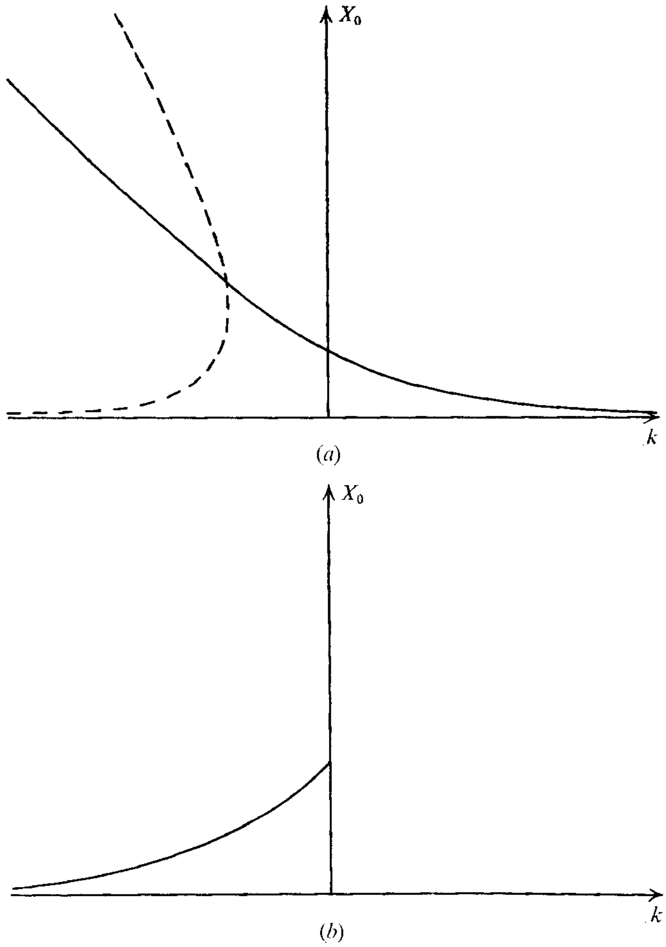

FIGURE 2. Response curve for equations (3.8a, b). (a) $G > 0$: ——, $GX_0^2 + 2kX_0 - V^2 = 0$; – – –, $GX_0^2 + 4kX_0 + 2V^2 = 0$. (b) $G < 0$: X_0 may take any value in $k > 0$.

shows that the only integral curve on which this assumption is satisfied is that which passes through the origin without being tangential to the Z axis there. The value of X at the intersection of this curve Γ with the initial surface $Y^2 + Z^2 = V^2X$ then gives the amplitude at the wave maker. We now briefly discuss this value as $\gamma \to 0$.

(i) $k < 0$, $G < 0$. For small γ, Γ is a regular perturbation about the curve

$$Y = 0, \quad Z^2 = -\tfrac{1}{2}GX^3 - 2kX^2, \quad Z < 0, \tag{3.10}$$

and so, as $\gamma \to 0$, the response is given by

$$GX_0^2 + 4kX_0 + 2V^2 = 0. \tag{3.11}$$

(ii) $k > 0$, $G > 0$. Γ is now a regular perturbation about the curve

$$Z = 0, \quad Y^2 = 2kX^2 + GX^3, \quad Y < 0, \tag{3.12}$$

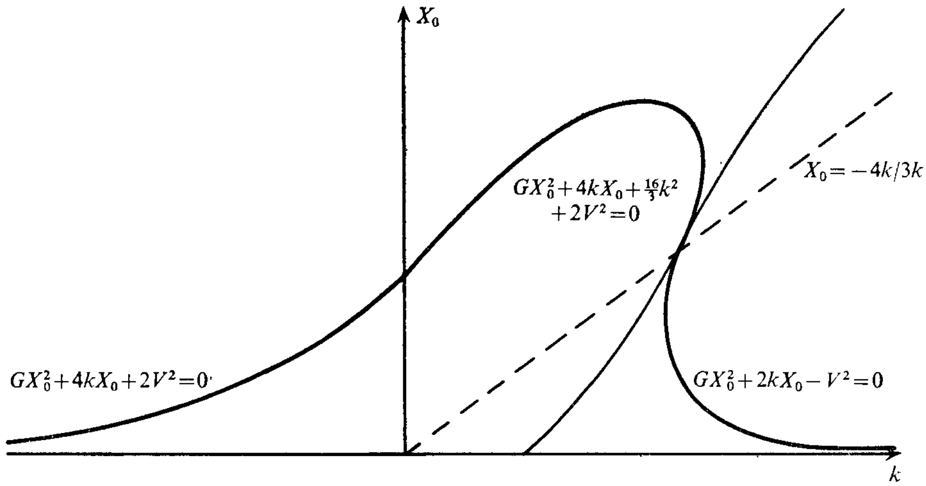

FIGURE 3. Response curve as $\gamma \to 0$ for $G < 0$.

so that the response is given by

$$GX_0^2+2kX_0-V^2 = 0. \tag{3.13}$$

Both (3.11) and (3.13) were suggested by the inviscid theory.

(iii) $k > 0$, $G < 0$. Γ may still be as in (ii) above as long as $k^2 > -GV^2$ and X_0 is a root of (3.13) which is less than $-4k/3G$. This restriction is necessary because perturbations about the curve (3.12) become singular near $X = -4k/3G$. If the initial surface does not intersect (3.12), Γ is more complicated and is shown in appendix B to consist of (B 6) for $Z > 0$, $X > -4k/3G$ and (3.12) for $Y < 0$, $X < -4k/3G$, joined by a transition region near $(-4k/3G, 0, 0)$. Thus, the response is any real root of

$$GX_0^2+4kX_0+16k^2/3G+2V^2 = 0 \tag{3.14}$$

as long as this root is greater than $-4k/3G$. We note that, in this second case, the distance taken for the wave train to decay is longer than if $k^2 > -GV^2$, since the transition region is of length of $O(\int Z^{-1}\,dX) \sim \gamma^{-\frac{1}{2}}$.

(iv) $k < 0$, $G > 0$. Γ is as in (i) above as long as $2k^2 > GV^2$, but if the initial surface does not intersect (3.10), Γ is again more complicated. In appendix C it is shown to consist initially of a series of loops near (3.10) and finally to be a regular perturbation about (3.12). These two regions are joined by a closely coiled spiral. The response curve is

$$GX_0^2+2kX_0-V^2 = 0 \tag{3.15}$$

and is that suggested by the inviscid theory.

The complete set of response curves is now as shown in figures 2(*a*) and 3. Without an appeal to stability theory, there is still non-uniqueness when G and k have opposite signs. An interesting feature of the curves for $G < 0$ is that the maximum attainable amplitude is finite and is quite independent of any damping mechanism. For $G > 0$, small non-zero viscosity does not prevent the amplitude becoming unbounded as $k \to -\infty$, although this happens on solution branches which may be unstable. The effect of viscosity on these solutions is not felt at amplitudes of $O(\epsilon^{-\frac{1}{2}})$.

4. Vertical oscillations of a cylindrical tank

In this section, we shall consider the motion of a cylindrical tank of liquid with vertical walls when it is subjected to vertical oscillations with amplitude ϵL and frequency ω, where L is a typical dimension of the tank. Experiment shows that for some values of ϵ and ω it is possible for the fluid to remain at rest relative to the container, whereas for other values surface waves whose frequency may be $\frac{1}{2}\omega$ or any integral multiple of ω are observed (Abramson 1966). The linear stability theory was analysed by Benjamin & Ursell (1954) and predicts an exponentially growing amplitude in the unstable cases. Here we shall discuss the effects of nonlinearity on the stability theory and in particular consider the possibility of obtaining a response of $O(\epsilon^{-1})$ compared with the forcing amplitude. For simplicity, the effects of viscosity are neglected throughout this section.

The equations of motion of the liquid referred to a suitable dimensionless co-ordinate system fixed in the tank are

$$\nabla^2\phi = 0, \tag{4.1}$$

$$\phi_y = 0 \quad \text{on} \quad y = -h, \tag{4.2}$$

$$\phi_n = 0 \quad \text{on the tank walls} \quad W(x,z) = 0 \tag{4.3}$$

and, on $y = \epsilon\eta$,

$$\phi_t + (g/L\omega^2 + \epsilon\cos t)\,\eta + \tfrac{1}{2}\epsilon|\nabla\phi|^2 = 0, \tag{4.4a}$$

$$\phi_y = \eta_t + \epsilon(\phi_x\eta_x + \phi_z\eta_z), \tag{4.4b}$$

where ϕ_n is the normal derivative of ϕ. The linear problem has the solution (Benjamin & Ursell 1954)

$$\phi = \sum_{m=1}^{\infty} \frac{\cosh k_m(y+h)}{k_m \sinh k_m h}\,\frac{da_m}{dt}\,S_m(x,z), \tag{4.5}$$

where k_m and S_m are the eigenvalues and eigenfunctions of the problem

$$S_{xx} + S_{zz} + k^2 S = 0$$

with

$$S_n = 0 \quad \text{on} \quad W(x,z) = 0,$$

and a_m satisfies

$$d^2a_m/dt^2 + k_m \tanh k_m h(g/\omega^2 L + \epsilon\cos t)\,a_m = 0. \tag{4.6}$$

The natural frequencies of the container are $\Omega_m = [gk_m \tanh(k_m h)/L]^{\frac{1}{2}}$ and (4.6) is a Mathieu equation for which the zero solution is unstable in certain regions of the $\epsilon L\Omega_m^2/g$, Ω_m^2/ω^2 plane.

In considering the effect of nonlinearity, we suppose that only the first mode of the system is excited appreciably and that this is achieved when $\Omega_1/\omega = \frac{1}{2} + \bar{\delta}$, where $\bar{\delta} \ll 1$. Linear stability theory predicts the local stability boundary as $|\bar{\delta}| = \epsilon\Omega_1^2 L/2g$ with exponential growth of small disturbances when $|\bar{\delta}|$ is less than this value. This result can be predicted by use of a multiple-scale perturbation to obtain a uniformly valid large-time expansion. However, we can still construct a uniformly valid approximation to the full equations of motion by writing

$$\phi \sim \epsilon^{-\frac{1}{2}}\phi_0 + \phi_1 + \epsilon^{\frac{1}{2}}\phi_2 + \ldots,$$

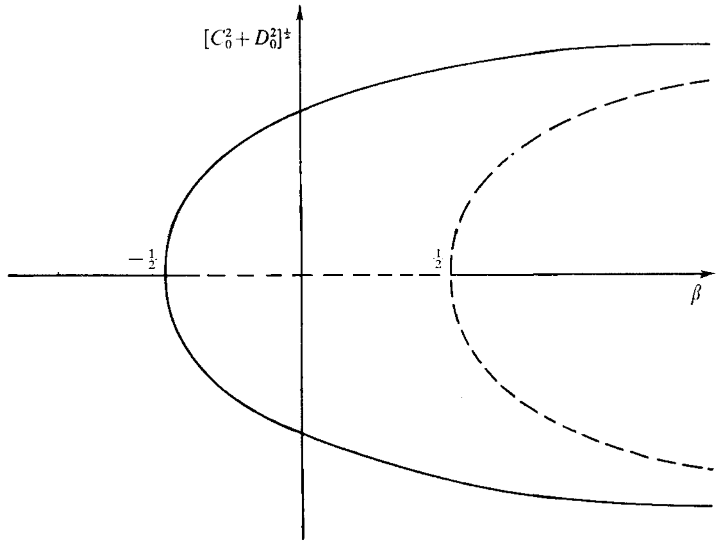

FIGURE 4. Bifurcation diagram for equation (4.9) when $t_1 > 0$. ——, stable solution; – – –, unstable solution.

where $\phi_i = \phi_i(x, y, z, t, \tau)$ and $\tau = \epsilon t$. Then

$$\phi_0 = (C_0(\tau)\cos\tfrac{1}{2}t + D_0(\tau)\sin\tfrac{1}{2}t)\frac{\cosh k_1(y+h)}{k_1\sinh k_1 h}S_1(x,z), \tag{4.7}$$

and ϕ_1 has an expansion similar to (4.5). Moreover, on $y = 0$

$$\phi_{2tt} + \mu\phi_{2y} = H_1\cos\tfrac{1}{2}t + H_2\sin\tfrac{1}{2}t + \text{higher harmonics in } t, \tag{4.8}$$

where $\mu^{-1} = 4k_1\tanh k_1 h = 4\Omega_1^2 L/g$ and

$$\frac{H_1}{C_0} - (\tfrac{1}{2}-\beta)S_1 + \frac{4\mu S_1}{C_0}\frac{dD_0}{d\tau} = \frac{H_2}{D_0} + (\tfrac{1}{2}+\beta)S_1 - \frac{4\mu S_1}{D_0}\frac{dC_0}{d\tau} = (C_0^2+D_0^2)P(x,z).$$

Here $\beta = 4\bar{\delta}\mu/\epsilon$ and $P(x,z) = \sum_{m=1}^{\infty} t_m S_m$ is a known function depending on ϕ_1. ϕ_2 is thus only bounded as $t \to \infty$ if

$$4\mu\, dC_0/d\tau = D_0[\tfrac{1}{2}+\beta - t_1(C_0^2+D_0^2)], \tag{4.9a}$$

$$4\mu\, dD_0/d\tau = C_0[\tfrac{1}{2}-\beta + t_1(C_0^2+D_0^2)]. \tag{4.9b}$$

A phase plane analysis of (4.9) (Struble 1962, p. 251) shows that if $t_1 > 0$ and $\beta < -\frac{1}{2}$ the only solution is $C_0 = D_0 = 0$ and this is a centre. However, for $|\beta| < \frac{1}{2}$ this solution becomes a saddle point and there exist two centres at $C_0 = 0$, $D_0 = \pm[(\beta+\frac{1}{2})/t_1]^{\frac{1}{2}}$. For $\beta > \frac{1}{2}$ the zero solution is again a centre, the above centres persist and there are saddle points at $C_0 = \pm[(\beta-\frac{1}{2})/t_1]^{\frac{1}{2}}$, $D_0 = 0$. The possible solution branches are illustrated for $t_1 > 0$ in the bifurcation diagram figure 4. If $t_1 < 0$, β is replaced by $-\beta$ in this diagram.

The effect of viscosity in the boundary layer on the base of the tank can be taken into account using the methods of §§ 2 and 3. This effect first influences ϕ_2 when $Re = O(\epsilon^{-2})$. Equations (4.9) then contain extra terms which are linear in C_0 and D_0 and these terms change the centres of (4.9) into stable spiral points.

The amplitude of the stable solution grows parabolically with β and no further scaling has been found with $\beta < O(\epsilon^{-1})$ which describes the subsequent growth of this solution. Indeed, if this growth persists until $\beta \sim O(\epsilon^{-1})$ so that $\phi \sim O(\epsilon^{-1})$ when $\bar{\delta} \sim O(1)$, in dimensional variables an $O(L)$ response is possible with an $O(\epsilon L)$ excitation. Such a solution would satisfy the full nonlinear equations (4.1)–(4.4) without the term $\epsilon \cos t\eta$ and would not be amenable to a perturbation analysis. An analogous situation occurs for the model equation

$$d^2y/dt^2 + (\tfrac{1}{4} + \bar{\delta})\, y + \epsilon(y \cos t + y^2) = 0. \tag{4.10}$$

An analysis of periodic solutions of this equation leads to equations identical to (4.9) when $\bar{\delta} \sim O(\epsilon)$. However, when $\bar{\delta} \sim O(1)$ the equation can be solved in terms of elliptic functions, and a solution of the type postulated above can be shown to exist.

A similar analysis can be carried out near the other frequencies for which the zero solution is unstable in linearized theory by writing $\Omega_1/\omega = n + \delta_n$, where n is an integer, and expanding appropriately. For example, when $n = 1$, we put $\phi \sim \phi_0 + \epsilon\phi_1 + \epsilon^2\phi_2 + \ldots$, and use the two time scales t and $\epsilon^2 t$, the region of interest being $\delta_1 \sim O(\epsilon^2)$. A bifurcation diagram can be obtained which is more involved than figure 4 when $\delta_1 \sim O(\epsilon^2)$ but, as $\delta_1/\epsilon^2 \to +\infty$, it resolves into the stable solution $A_0 = B_0 = 0$ and two other branches with amplitude of $O(\delta_1^{\frac{1}{2}}/\epsilon)$ of which one is stable and the other unstable.

The authors are grateful for the hospitality of the Courant Institute of Mathematical Sciences, New York University, where much of this work was carried out. The research was supported by the U.S. Army Research Office-Durham, nnder Grant No. DA-ARO-D-31-124-72-G47.

Appendix A

We shall not attempt an exhaustive numerical or asymptotic analysis of the continuous, periodic, zero-mean solutions of (2.11), but merely describe the qualitative behaviour of some solutions with these properties. This will be done by means of the schematic response diagrams for the distance from crest to trough as a function of λ in figures 5 and 6. The shape of these response diagrams has been determined more accurately by computation using the truncated Fourier series method described by Chester (1968).

The asymptotic behaviour for large κ is easiest to describe and for $\lambda \sim \kappa^{-\frac{2}{3}} \ll 1$, one solution has been shown in § 2 to match with that for $h = O(1)$, with a response curve as in figure 1. As λ increases with $\kappa^2 \gg |\lambda| \gg \kappa^{-\frac{2}{3}}$, the growing and decaying branches of figure 1 evolve into expansions of the form $g \sim \pm\frac{1}{3}\kappa\lambda^{\frac{1}{2}}2^{\frac{3}{2}}\cos t$ and $g \sim (2/\lambda\pi)\cos t$ respectively. When $\bar{\kappa} = \kappa^2/\lambda = O(1)$, the first of these expansions becomes $g \sim \lambda g_0 + \ldots$, where g_0 satisfies (2.11) without the forcing term $(2/\pi)\cos t$ and can be written down in terms of elliptic functions. The second expansion is still valid except near the points $\bar{\kappa} = 3/(1-N^2)$, where N is an integer greater than one. For $N > 2$, as λ decreases through these points, g_0 passes from one solution branch to another and is locally modified by the addition of harmonics with period $2\pi/N$. However, for $N = 2$, the expansion itself is modified locally

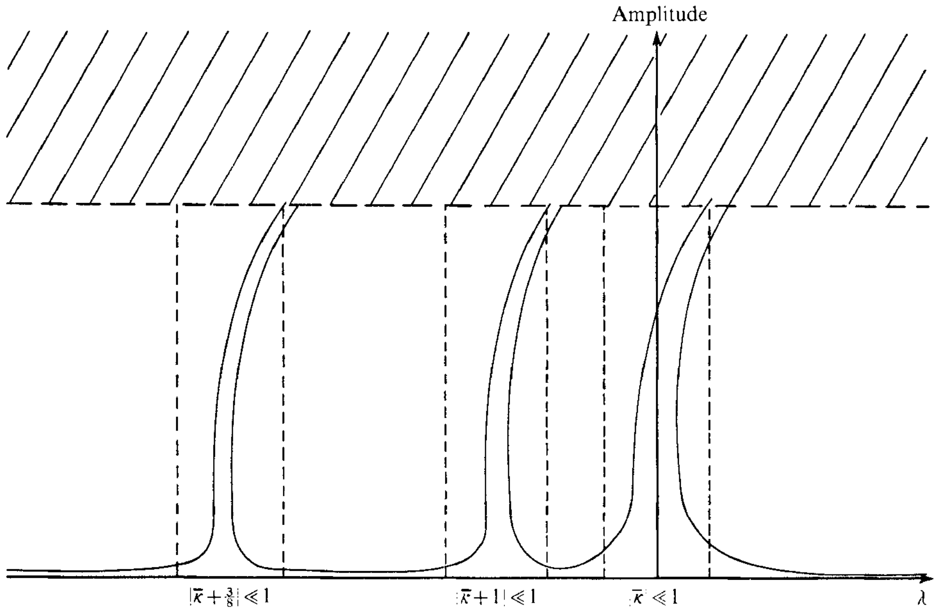

FIGURE 5. Response curves for equation (2.11) when $\kappa \gg 1$, $\bar{\kappa} = O(1)$.

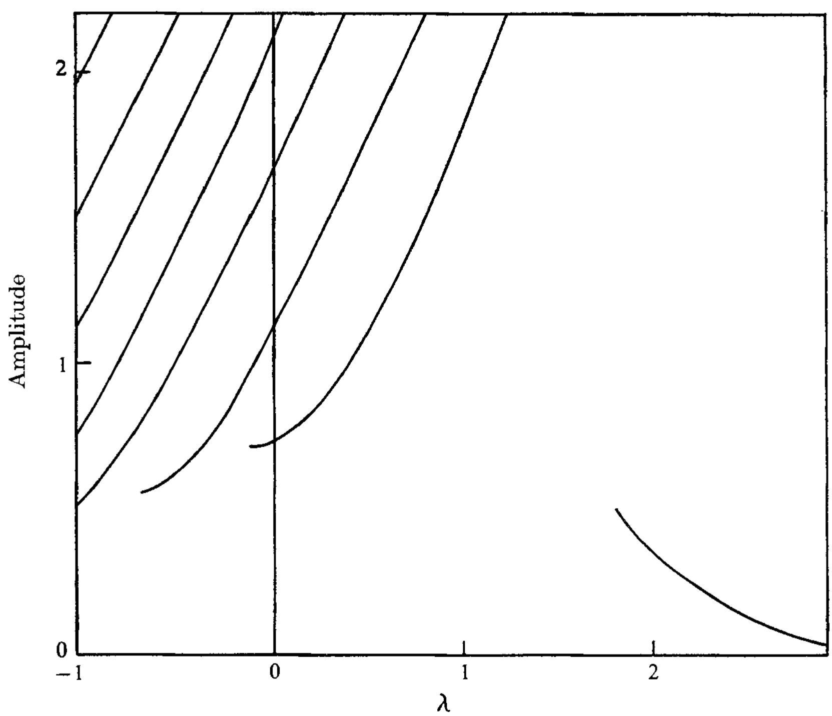

FIGURE 6. Response curves for equation (2.11) when $\kappa = 0{\cdot}1$.

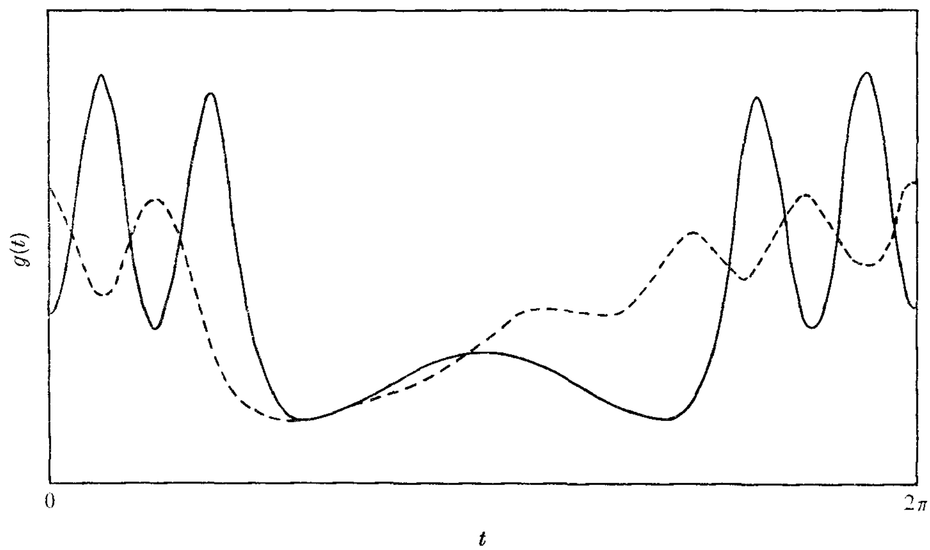

FIGURE 7. Surface elevations for $\kappa = 0{\cdot}1$; $\lambda = -0{\cdot}5$. ——, $\mu = 0$; – – –, $\mu = 0{\cdot}5$.

to $g \sim \lambda^{-\frac{1}{2}}\bar{g}_0 + \ldots$, where $\bar{g}_0$ is periodic with period π. Moreover, near any of these points, the magnitude of g may grow from $O(\lambda^{-1})$ to $O(\lambda)$, so that the response curve for $\bar{\kappa} = O(1)$ is as shown in figure 5. The shaded region indicates where the solution is in principle described by elliptic functions.

Although equation (2.11) has many intriguing asymptotic properties, we shall not go into any further detail because the effect of even a small viscosity coefficient μ in (2.13) rapidly prevents the high frequency branches with large values of N in figure 5 from attaining more than a small amplitude.

Finally, the results of the inviscid numerical solutions when κ is small and $\lambda = O(1)$ are shown in figure 6. For $\kappa = 0{\cdot}1$, computation was only possible for $\lambda > -1$ because of the very high frequency oscillations which occur for smaller values of λ. These results suggest that, while there is no continuous solution for $\kappa = 0$ and $|\lambda| < 1{\cdot}76$ (see § 2), the effect of arbitrarily small dispersion is to produce a large number of highly oscillatory solutions for small $\kappa \neq 0$. Again, we shall not discuss this conjecture further here because, as has been suggested by Chester (1968), the most physically reasonable solution when $\kappa = 0$ has a compressive discontinuity and this is probably also true for small non-zero κ. Indeed, the numerical solution for small μ in this case not only diminishes the amplitude but gives an asymmetrical surface elevation with a smoothed compressive discontinuity. This may be seen from the typical elevations with and without viscosity present shown for $\kappa = 0{\cdot}1$ in figure 7. Figure 7 may be compared with the experimental results of Chester & Bones (1968), whose own figure 7 is for comparable values of the parameters.

Appendix B

As ξ decreases from $+\infty$, Γ begins at the origin and lies near (3.12) for

$$X < -4k/3G.$$

Integral curves which are near (3.12) for small γ do not however remain so uniformly near the point $(-4k/3G,\ 2^{\frac{5}{2}}k^{\frac{3}{2}}/3^{\frac{3}{2}}G,\ 0)$, where their behaviour may be described by putting

$$X+\frac{4k}{3G} = -\frac{\gamma^{\frac{2}{5}}2^{\frac{9}{5}}k^{\frac{3}{5}}}{3^{\frac{6}{5}}G}\,X', \tag{B 1}$$

$$Y-\frac{2^{\frac{5}{2}}k^{\frac{3}{2}}}{3^{\frac{3}{2}}G} = \frac{\gamma^{\frac{4}{5}}2^{\frac{11}{10}}k^{\frac{7}{10}}}{3^{\frac{9}{10}}G}\,Y', \tag{B 2}$$

$$Z = \frac{\gamma^{\frac{3}{5}}2^{\frac{17}{10}}k^{\frac{9}{10}}}{3^{\frac{13}{10}}G}\,Z'. \tag{B 3}$$

Then, to first order

$$Z'\,dZ'/dX' = Y'+X'^2, \quad Z'\,dY'/dX' = 1. \tag{B 4}$$

We require a transition solution of these equations which matches with an integral curve near (3.12). An expansion of (3.12) thus gives the matching conditions as

$$Y' \sim -X'^2, \quad Z' \sim -(2X')^{-1} \quad \text{as} \quad X' \to -\infty. \tag{B 5}$$

Now the further transformation $Z' = dX'/du$ gives, apart from an arbitrary translation of u,

$$d^2X'/du^2 = u+X'^2,$$

so that $X'(u)$ is the Painlevé transcendent (Davis 1962, p. 229). Hence the requirement that $X' \sim -(-u)^{\frac{1}{2}}$ as $u \to -\infty$ determines X' uniquely as a monotonic increasing function with $X' \sim 6(u-3{\cdot}41)^{-2}$ near $u = 3{\cdot}41$. Thus, as $X' \to +\infty$, $Z' \sim 2^{\frac{1}{2}}3^{-\frac{1}{2}}X'^{\frac{3}{2}}$ and the transition solution matches with an integral curve near

$$Z^2 = -\tfrac{1}{2}G(X+4k/3G)^3, \quad Y = 2^{\frac{5}{2}}k^{\frac{3}{2}}/3^{\frac{3}{2}}G. \tag{B 6}$$

Appendix C

It is most convenient to consider the projection of Γ on $Z = 0$ which is the solution of

$$\frac{\gamma^2}{2}\frac{d}{dX}\left[X\left(\frac{dX}{dY}\right)^2\right]+4k+2GX = \frac{2Y^2}{X^2} \tag{C 1}$$

with $X = 0$ and dX/dY finite at $Y = 0$. As in appendix B we consider $X(Y)$ as ξ decreases from $+\infty$. This means that Y remains negative throughout. For Y of $O(\gamma)$ the curve (C 1) is oscillatory and basically consists of parabolic loops. In this region Γ lies uniformly near $Z^3 = -\frac{1}{2}GX^3-2kX^2$. For larger values of Y we use the method of multiple scales with a stretched slow variable (Cole 1968, p. 103) and define fast and slow variables by $\overline{Y} = -\gamma^{-1}Y$ and $\tilde{Y} = -\gamma^{-\frac{1}{3}}Y$ respectively. We also put $dY^*/d\overline{Y} = \psi(\tilde{Y})$, where ψ is determined by the condition that the period T of the rapidly varying part of the solution should be independent of the slow variable. Expanding $X \sim X_0(\tilde{Y}, Y^*) + \gamma^{\frac{2}{3}} X_1(\tilde{Y}, Y^*) + \dots$ then gives

$$\tfrac{1}{2}\psi^2 X_0 (\partial X_0/\partial Y^*)^2 + 4kX_0 + GX_0^2 = C(\tilde{Y}), \tag{C 2}$$

where C is as yet arbitrary except that $-4k^2/G \leqslant C \leqslant 0$. X_1 then satisfies a linear non-homogeneous differential equation, one of whose complementary functions is $\partial X_0/\partial Y^*$. One condition for the boundedness of X_1 is that this complementary function should be orthogonal to the non-homogeneous term, which gives

$$\int_0^T \frac{\partial X_0}{\partial Y^*}\left[X_0 \frac{d\psi}{d\tilde{Y}}\frac{\partial X_0}{\partial Y^*} + 2\psi X_0 \frac{\partial^2 X_0}{\partial \tilde{Y}\,\partial Y^*} + \psi \frac{\partial X_0}{\partial Y^*}\frac{\partial X_0}{\partial \tilde{Y}} - 2\frac{\tilde{Y}^2}{X_0^2}\right] dY^* = 0.$$

Using (C 2), this may be rewritten as

$$\frac{d}{d\tilde{Y}}\left[\psi \int_0^T X_0 \left(\frac{\partial X_0}{\partial Y^*}\right)^2 dY^*\right] = 2\tilde{Y}^2 \int_0^T \frac{\partial X_0}{\partial Y^*}\frac{dY^*}{X_0^2},$$

or

$$\frac{d}{d\tilde{Y}}\int_{\alpha_1}^{\alpha_2} -[-GX_0^3 - 4kX_0^2 + CX_0]^{\frac{1}{2}}\, dX_0 = \frac{2^{\frac{1}{2}}(\alpha_2 - \alpha_1)\,\tilde{Y}^2}{\alpha_1 \alpha_2},$$

where $\alpha_{1,2}(\tilde{Y}) = -2k/G \mp [4k^2/G^2 + C(\tilde{Y})/G]^{\frac{1}{2}}$ respectively. Finally we obtain

$$\tilde{Y}^2 \frac{d\tilde{Y}}{d\chi} = \frac{1}{4\chi}\left(\chi^2 - \frac{4k^2}{G}\right)\frac{dI}{d\chi}, \tag{C 3}$$

where $\chi = [4k^2/G^2 + C(\tilde{Y})/G]^{\frac{1}{2}}$ and

$$I = (2G)^{\frac{1}{2}} \int_{-\chi - 2k/G}^{\chi - 2k/G} \left[\left(\chi - \frac{2k}{G} - X_0\right)\left(\chi + \frac{2k}{G} + X_0\right) X_0\right]^{\frac{1}{2}} dX_0.$$

$C(\tilde{Y})$ is determined by (C 3) and, if needed, ψ is given by the condition that T is independent of $\tilde{Y}$. This solution matches with that for $Y = O(\gamma)$ if $C \to 0$ as $\tilde{Y} \to 0$, in which case $\tilde{Y} = 0$ when $\chi = -2k/G$. Consideration of (C 3) then shows that $C \to -4k^2/G$ and $\chi \to 0$ for some finite value of $\tilde{Y}$, say $\tilde{Y}_0$. Furthermore $\chi = O(\tilde{Y} - \tilde{Y}_0)$ as $\tilde{Y} \to \tilde{Y}_0$ and, by appropriately expanding the elliptic functions in the solution of (C 2), we can show that

$$X_0 \sim -\frac{2k}{G} + O\left\{(\tilde{Y} - \tilde{Y}_0)\cos\frac{GY^*}{(-k)^{\frac{1}{2}}\psi(\tilde{Y})}\right\} \tag{C 4}$$

as $\tilde{Y} \to \tilde{Y}_0$. Thus our transition solution matches to one term with the outer solution, valid for $Y = O(1)$, whose first term is (3.12). However, the form of (C 4) suggests that matching to higher order would be a complicated matter.

REFERENCES

ABRAMSON, H. N. (ed.) 1966 *The Dynamic Behaviour of Liquids in Moving Containers*. N.A.S.A.: SP-106.

BENJAMIN, T. B. & URSELL, F. 1954 *Proc. Roy. Soc.* A **225**, 505.

CHESTER, W. 1964 *J. Fluid Mech.* **18**, 44.

CHESTER, W. 1968 *Proc. Roy. Soc.* A **306**, 5.

CHESTER, W. & BONES, J. A. 1968 *Proc. Roy. Soc.* A **306**, 23.

COLE, J. D. 1968 *Perturbation Methods in Applied Mathematics*. Waltham, Mass.: Blaisdell.

COLLINS, W. 1971 *Quart. J. Mech. Appl. Math.* **24**, 129.

DAVIS, H. T. 1962 *Introduction to Nonlinear Differential and Integral Equations*. Dover.

FULTZ, D. 1962 *J. Fluid Mech.* **13**, 193.

MAHONY, J. J. 1971 *Fluid Mech. Res. Inst. University of Essex Rep.* no. 7.

MILES, J. W. 1967 *Proc. Roy. Soc.* A **297**, 459.

MOISEYEV, N. N. 1958 *Prikl. Mat. Mekh.* **22**, 612.

STRUBLE, R. A. 1962 *Nonlinear Differential Equations*. McGraw-Hill.

TADJBAKHSH, I. & KELLER, J. B. 1960 *J. Fluid Mech.* **8**, 442.

TAYLOR, G. I. 1953 *Proc. Roy. Soc.* A **218**, 44.

URSELL, F. 1952 *Proc. Roy. Soc.* A **214**, 79.